SOLUTION OF THE INVERSE PROBLEM OF THERMAL CONDUCTIVITY FOR A MELTING SLAB WITH ENTRAINMENT

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This article proposes an approximate solution to the inverse problem of the Stefan type for a finite region with arbitrary boundary and initial conditions. A comparison with exact solutions is made.

The inverse problem of thermal conductivity for a region with a movable boundary consists in finding the law governing the motion of the boundary of a melting solid $s(\tau)$ and of the heat fluxes $q_1(\tau)$ and $q_2(\tau)$ on the basis of a known change in the temperature at two external points, x_1 and x_2, of the slab, $t_1(x, \tau)$ and $t_2(x, \tau)$. We shall assume that these temperatures are measured experimentally without systematic errors. There is no general method for solving problems of this type. Work is known [1, 2] in which the inverse problem of thermal conductivity is solved with a given law of motion of the phase interface. In practice, we are forced to face the necessity of solving the inverse problem of thermal conductivity with an unknown law of motion of the interface.

In the present article an inverse problem of the Stefan type is solved for a finite region with arbitrary boundary and initial conditions using the method of successive intervals [3]. The temperature $t(x, \tau)$ at any arbitrary point of the slab satisfies the equation of thermal conductivity

$$\frac{\partial^2 t(x,\tau)}{\partial x^2} = \frac{1}{a}\frac{\partial t(x,\tau)}{\partial \tau} \qquad (0 \leqslant x \leqslant s(\tau)) \tag{1}$$

with the boundary conditions

$$\lambda\, \partial t(s, \tau)/\partial x = q_1(\tau) + \rho L\, ds/d\tau \tag{2}$$

$$\lambda\, \partial t(0, \tau)/\partial x = q_2(\tau) \tag{3}$$

$$t(s, \tau) = T \tag{4}$$

$$s(\tau_m) = R \tag{5}$$

Here λ and a are the coefficients of thermal conductivity and thermal diffusivity of the substance of the slab; ρ is its density; T is the melting temperature; L is the specific heat of fusion or the effective heat of fusion, taking account also of the heat of the chemical reactions, taking place at a constant temperature T; R is the original thickness of the slab; $q_1(\tau)$ and $q_2(\tau)$ are the heat fluxes at the boundaries of the region $x=s(\tau)$ and $x=0$, respectively. It is assumed that the melt is removed instantaneously by mechanical action.

If the initial distribution of the temperature is approximated by a polynomial of the fourth order

$$t(x,0) = \varphi(x) = A + Bx/R + C(x/R)^2 + D(x/R)^3 + E(x/R)^4 \tag{6}$$

the solution of the direct problem (1)-(6) can be represented in the form [3]

$$t(x, \omega) = t(x, 0) + BF\left(\frac{x}{R}, \omega\right) - (B + 2C + 3D + 4E)\, F\left(\frac{R-x}{R}, \omega\right) +$$

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$$+6D\Phi\left(\frac{x}{R}, \omega\right)-(6D+24E)\Phi\left(\frac{R-x}{R}, \omega\right)+12E\omega^2+\left(2C+6D\frac{x}{R}+12E\frac{x^2}{R^2}\right)\omega+$$
$$+\frac{R}{\lambda}\left[\sum_{i=0}^{N} Q_{1,i+1}\Delta F\left(\frac{R-x}{R}, \omega-i\Delta\omega\right)-\sum_{i=0}^{N} q_{2,i+1}\Delta F\left(\frac{x}{R}, \omega-i\Delta\omega\right)\right] \quad (7)$$
$$\omega=\frac{a\tau}{R^2}, \quad \Delta\omega=\frac{a\Delta\tau}{R^2}, \quad N\Delta\omega\leqslant\omega\leqslant(N+1)\Delta\omega$$
$$Q_{1,i+1}\equiv q_{1,i+1}, \quad \tau<\tau_m, \quad Q_{1,i+1}\equiv q_{1,i+1}+q_{1f,i+1}, \quad \tau>\tau_m$$

where $q_{i,\,i+1}$ and $q_{1f,\,i+1}$ are the values of the quasi-constant real and fictitious heat fluxes for the (i + 1)-th interval of time, with stepwise approximation of the heat fluxes [3]; the functions $F(y, \omega)$ and $\Phi(y, \omega)$ were found in [3] and have the form

$$F(y, \omega)=2\sqrt{\omega}\sum_{k=0}^{\infty}\left\{\operatorname{ierfc}\left(\frac{2k+y}{2\sqrt{\omega}}\right)+\operatorname{ierfc}\left[\frac{2(k+1)-y}{2\sqrt{\omega}}\right]\right\} \quad (8)$$

$$\Phi(y, \omega)=8\omega\sqrt{\omega}\sum_{k=0}^{\infty}\left\{\mathrm{i}^3\operatorname{erfc}\left(\frac{2k+y}{2\sqrt{\omega}}\right)+\mathrm{i}^3\operatorname{erfc}\left[\frac{2(k+1)-y}{2\sqrt{\omega}}\right]\right\}$$
$$\Delta F(y, n\Delta\omega)=F(y, n\Delta\omega)-F[y, (n-1)\Delta\omega] \quad (9)$$

Solving a system of two equations of the type of (7) for $x = x_1$ and $x = x_2$ with respect to $Q_{1,\,N+1}$ and $q_{2,\,N+1}$, we obtain

$$Q_{1,N+1}=\|D\|^{-1}\left\{\left[\frac{\lambda}{R}\Delta t_\varphi(x_1, (N+1)\Delta\omega)-\sum_{i=0}^{N-1} Q_{1,i+1}\Delta F\left(1-\frac{x_1}{R}, (N+1-i)\Delta\omega\right)+\sum_{i=0}^{N-1} q_{2,i+1}\Delta F\left(\frac{x_1}{R}, (N+1-i)\Delta\omega\right)\right]\Delta F\left(\frac{x_2}{R}, \Delta\omega\right)-\right.$$
$$-\left[\frac{\lambda}{R}\Delta t_\varphi(x_2, (N+1-i)\Delta\omega)-\sum_{i=0}^{N-1} Q_{1,i+1}\Delta F\left(1-\frac{x_2}{R}, (N+1-i)\Delta\omega\right)+\right.$$
$$\left.\left.+\sum_{i=0}^{N-1} q_{2,i+1}\Delta F\left(\frac{x_2}{R}, (N+1-i)\Delta\omega\right)\Delta F\left(\frac{x_1}{R}, \Delta\omega\right)\right\} \quad (10)$$

$$q_{2,N+1}=\|D\|^{-1}\left\{\left[\frac{\lambda}{R}\Delta t_\varphi(x_1, (N+1)\Delta\omega)-\sum_{i=0}^{N-1} Q_{1,i+1}\times\right.\right.$$
$$\times\left.\Delta F\left(1-\frac{x_1}{R}, (N+1-i)\Delta\omega\right)+\sum_{i=0}^{N-1} q_{2,i+1}\Delta F\left(\frac{x_1}{R}, (N+1-i)\Delta\omega\right)\right]\times$$
$$\times\Delta F\left(1-\frac{x_2}{R}, (N+1-i)\Delta\omega\right)-$$
$$-\left[\frac{\lambda}{R}\Delta t_\varphi(x_2, (N+1)\Delta\omega)-\sum_{i=0}^{N-1} Q_{1,i+1}\Delta F\left(1-\frac{x_2}{R}, (N+1-i)\Delta\omega\right)+\right.$$
$$\left.\left.+\sum_{i=0}^{N-1} q_{2,i+1}\Delta F\left(\frac{x_2}{R}, (N+1-i)\Delta\omega\right)\right]\Delta F\left(1-\frac{x_1}{R}, (N+1-i)\Delta\omega\right)\right\} \quad (11)$$

$$\|D\|=\Delta F\left(1-\frac{x_1}{R}, \Delta\omega\right)\Delta F\left(\frac{x_2}{R}, \Delta\omega\right)-\Delta F\left(1-\frac{x_2}{R}, \Delta\omega\right)\Delta F\left(\frac{x_1}{R}, \Delta\omega\right)$$
$$\Delta t_\varphi(x, \omega)=t(x, \omega)-t(x, 0)-BF\left(\frac{x}{R}, \omega\right)+$$
$$+(B+2C+3D+4E)F\left(1-\frac{x}{R}, \omega\right)-6D\Phi\left(\frac{x}{R}, \omega\right)+$$
$$+(6D+24E)\Phi\left(1-\frac{x}{R}, \omega\right)-12E\omega^2-(2C+6DxR^{-1}+12Ex^2R^{-2})\omega \quad (12)$$

If the surface $x = 0$ is heat-insulated $[q_2(\tau)\equiv 0]$, the expression for $Q_1(\tau)$ is simplified:

$$Q_{1,N+1}=\left[\Delta F\left(1-\frac{x_1}{R}, \Delta\omega\right)\right]^{-1}\left\{\frac{\lambda}{R}\Delta t_\varphi[x_1, (N+1)\Delta\omega]-\sum_{i=0}^{N-1} Q_{1,i+1}\Delta F\left[1-\frac{x_1}{R}, (N+1-i)\Delta\omega\right]\right\} \quad (13)$$

Relationships (10), (11) permit determining the unknown heat flux $q_2(\omega)$ from values of $q_{2,N+1}$ corresponding to the mean point of the interval $[N\Delta\omega, (N=1)\Delta\omega]$, i.e., the point $(N+1/2)\Delta\omega$.

To determine the law of change in the boundary of the melting solid, in relationship (7) we set $x=R\cdot(1-\xi_{N+1})$, where $1-s(\omega)/R=\xi(\omega)$ is the relative depth of the melting. Taking account of condition (4), we obtain

$$\begin{aligned}&A+B(1-\xi_{N+1})+C(1-\xi_{N+1})^2+D(1-\xi_{N+1})^3+E(1-\xi_{N+1})^4+\\&+2C(N+1)\Delta\omega+6D(N+1)\Delta\omega(1-\xi_{N+1})+12E(N+1)\Delta\omega(1-\xi_{N+1})^2=\\&=T-BF[1-\xi_{N+1},(N+1)\Delta\omega]+(B+2C+3D+4E)\times\\&\times F[\xi_{N+1},(N+1)\Delta\omega]-6D\Phi[1-\xi_{N+1},(N+1)\Delta\omega]+\\&+(6D+24E)\Phi[\xi_{N+1},(N+1)\Delta\omega]-12E(N+1)^2\Delta\omega^2-\\&-\frac{R}{\lambda}\left\{\sum_{i=0}^{N}Q_{1,i+1}\Delta F[\xi_{N+1},(N+1-i)\Delta\omega]-\sum_{i=0}^{N}q_{2,i+1}\Delta F\left[1-\xi_{N+1},(N+1-i)\Delta\omega\right]\right\}\end{aligned}\tag{14}$$

Removing the brackets on the right-hand side of Eq. (14), replacing the terms ξ^3_{N+1} and ξ^4_{N+1} by $\xi^2_{N+1}\xi_N$ and $\xi^2_{N+1}\xi_N^2$ and the arguments ξ_{N+1} in the functions F, ΔF, and Φ on the left-hand side of (14) by ξ_N, we obtain a quadratic equation for ξ_{N+1}:

$$\begin{aligned}&[C+3D+6E+12E(N+1)\Delta\omega-D\xi_N-4E\xi_N+E\xi_N^2]\,\xi^2_{N+1}-\\&-[B+2C+3D+4E+6(N+1)\Delta\omega+24E(N+1)\Delta\omega]\,\xi_{N+1}-\\&-\{T-(A+B+C+D+E)-(2C+6D+12E)(N+1)\Delta\omega-\\&-12E(N+1)^2\Delta\omega^2-BF[1-\xi_N,(N+1)\Delta\omega]+(B+2C+3D+4E)\times\\&\times F[\xi_N,(N+1)\Delta\omega]-6DF[1-\xi_N,(N+1)\Delta\omega]+\\&+(6D+24E)\Phi[\xi_N,(N+1)\Delta\omega]-\frac{R}{\lambda}\sum_{i=0}^{N}Q_{1,i+1}\Delta F[\xi_N,(N+1-i)\Delta\omega]+\\&+\frac{R}{\lambda}\sum_{i=0}^{N}q_{2,i+1}\Delta F[1-\xi_N,(N+1-i)\Delta\omega)]\}=0\end{aligned}$$

The heat flux $q_1(\omega)$ is determined using the Stefan condition (2):

$$q_1(\omega)=\lambda\frac{\partial t}{\partial x}+\rho\frac{aL}{R}\frac{d\xi}{d\omega}=\lambda\frac{\partial t}{\partial x}+\frac{\lambda L}{cR}\frac{d\xi}{d\omega},\quad x=R(1-\xi)\tag{15}$$

Differentiating expression (7) with respect to x, then substituting $x=R(1-\xi)$ and replacing the derivative $d\xi/d\omega$ by the finite difference $(\xi_{N+1}-\xi_N)/\Delta\omega$, we obtain from (15)

$$\begin{aligned}&q_{1,N+1}=\frac{\lambda L}{cR}\frac{\xi_{N+1}-\xi_N}{\Delta\omega}+\frac{\lambda}{R}\{B+2C(1-\xi_{N+1})+3D(1-\xi_{N+1})^2+\\&+4E(1-\xi_{N+1})^3-B\vartheta[1-\xi_{N+1},(N+1)\Delta\omega]-(B+2C+3D+4E)\times\\&\times\vartheta[\xi_{N+1},(N+1)\Delta\omega]-6D\vartheta_1[1-\xi_{N+1},(N+1)\Delta\omega]-\\&-(6D+24E)\vartheta_1[\xi_{N+1},(N+1)\Delta\omega]+6D(N+1)\Delta\omega+\\&+24E(N+1)\Delta\omega(1-\xi_{N+1})\}+\sum_{i=0}^{N}Q_{1,i+1}\Delta\vartheta[\xi_{N+1},(N+1-i)\Delta\omega]+\\&+\sum_{i=0}^{N}q_{2,i+1}\Delta\vartheta[1-\xi_{N+1},(N+1-i)\Delta\omega]\end{aligned}\tag{16}$$

$$\vartheta_1(y,\omega)=4\omega\sum_{k=0}^{\infty}\left\{\mathrm{i^2erfc}\left(\frac{2k+y}{2\sqrt{\omega}}\right)-\mathrm{i^2erfc}\left[\frac{2(k+1)-y}{2\sqrt{\omega}}\right]\right\}$$

$$\vartheta(y,\omega)=\sum_{k=0}^{\infty}\left\{\mathrm{erfc}\left(\frac{2k+y}{2\sqrt{\omega}}\right)-\mathrm{erfc}\left[\frac{2(k+1)-y}{2\sqrt{\omega}}\right]\right\}$$

To evaluate the accuracy of the method set forth above, we give a numerical example for which exact solutions are also known.

The exact partial solution of the Sanders equation [4] has the form

$$q_2(\omega)=0$$

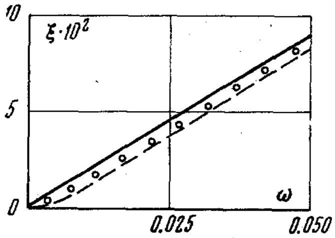

Fig. 1

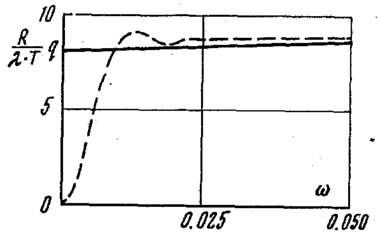

Fig. 2

$$t(x, \omega) = T(x^2R^{-2} + 2\omega) \tag{17}$$

$$\xi(\omega) = 1 - \sqrt{1-2\omega} \tag{18}$$

$$q_1(\omega) = \frac{\lambda T}{R}\left(\frac{L}{cT}\frac{1}{\sqrt{1-2\omega}} + 2\sqrt{1-2\omega}\right) \tag{19}$$

In this case, the initial distribution of the temperature is equal to

$$t(x, 0) = Tx^2R^{-2}$$

i.e., A = B = D = E = 0, C = T.

On the basis of values of the temperature measured at the point $x_1 = 0.9$ R, we can establish the law of motion of the boundary (18) and the heat flux (19). In this case, in accordance with (12)

$$\Delta t_\varphi(x_1, \omega) = 2TF(0.1; \omega)$$

Calculation of $Q_{1,N+1}$ using (13) at $x = s = R(1-\xi)$ gives $Q_{1,N+1} = 2\lambda T/R$, and from (14) we obtain an equation for $\xi(\omega)$:

$$[1-\xi(\omega)]^2 + 2\omega = 1 + 2F[\xi(\omega), \omega] - 2\sum_{i=0}^{N}\Delta F[\xi(\omega), (N+1-i)\Delta\omega] \tag{20}$$

By the definition of ΔF

$$\sum_{i=0}^{N}\Delta F[\xi(\omega), (N+1-i)\Delta\omega] \equiv \sum_{k=1}^{N+1}\Delta F[\xi(\omega), k\Delta\omega] \equiv F[\xi(\omega), \omega] \tag{21}$$

From (20) we obtain the equation

$$\xi^2(\omega) - 2\xi(\omega) + 2\omega = 0$$

whose solution is $\xi(\omega) = 1 - \sqrt{1-2\omega}$, which coincides with (18). From (16) we find an expression for the heat flux

$$q_1(\omega) = \frac{\lambda L}{cR}\frac{d\xi}{d\omega} + \frac{2\lambda T}{R}[1-\xi(\omega)] - \frac{2\lambda T}{R}\vartheta[\xi(\omega), \omega] + \frac{2\lambda T}{R}\sum_{i=0}^{N}\Delta\vartheta[\xi(\omega), (N+1-i)\Delta\omega]$$

Transforming the sum in the last term, similarly to (21) we obtain

$$q_1(\omega) = \frac{\lambda L}{cR}\frac{d\xi}{d\omega} + \frac{2\lambda T}{R}[1-\xi(\omega)] = \frac{\lambda T}{R}\left(\frac{L}{cT}\frac{1}{\sqrt{1-2\omega}} + 2\sqrt{1-2\omega}\right)$$

which coincides with (19). In this case the inverse problem is considerably simplified due to the constant nature of $Q_{1,N+1}$, which made it possible to carry out a summation using (21).

Another exact solution of the Sanders equation [4] has the form

$$\xi(\omega) = 1 - \sqrt{1 - 3.41612\omega} \tag{22}$$

$$t(x, \omega) = T\left\{1 - \sqrt{1-3.41612\omega}\; {}_1F_1\left(-\frac{1}{2}; \frac{1}{2}; \frac{0.85403x^2R^{-2}}{1-3.41612\omega}\right)\right\} \tag{23}$$

where ${}_1F_1(\alpha; \beta; z)$ is a degenerate hypergeometric function.

From condition (2) we obtain an expression for the heat flux

$$q_1(\omega) = \frac{\lambda T}{R}\left(2.34909 + \frac{1.70806}{\sqrt{1-3.41612\omega}}\,\frac{L}{cT}\right)$$

or, setting $L/cT = 10/3$ (as in the numerical example of Sanders)

$$q_1(\omega) = \frac{\lambda T}{R}\left(2.34909 + \frac{5.69353}{\sqrt{1-3.41612\omega}}\right) \tag{24}$$

In this case the initial distribution has the form

$$t(x, 0) = T\left[0.85403\,\frac{x^2}{R^2} + \frac{(0.85403)^2}{6}\,\frac{x^4}{R^4} + \cdots\right]$$

and, consequently,

$$4 = B = D = 0, \quad C = 0.85403T, \quad E = 0.12156T$$

If as starting information we take the temperatures (23) arising at the point $x_1 = 0.9R$, then, using formulas (14) and (16) we can find the depth of melting $\xi(\omega)$ and $q_1(\omega)$. Figures 1 and 2 give the results of a solution of the inverse problem with $\Delta\omega = 0.005$. The solid lines show exact solutions of (22) for $\xi(\omega)$ and of (24) for $q_1(\omega)$. The dotted lines show the functions ξ and q_1, obtained from (14) and (16), respectively. Values of ξ obtained by a shift of the argument by $\Delta\omega/2$ are plotted on Fig. 1 by the small circles.

It can be seen from Fig. 1 that the law for the shift of the boundary was correctly established. A systematic shift of the curve by approximately 0.5–$0.75\Delta\omega$ is observed. This shift is due to the insufficient accuracy of the iteration process [the replacement of ξ_{N+1} by ξ_N in the right-hand side of (14)]. Some improvement of the iteration process can be obtained by comparing the exact value of ξ_{N+1} not with the ξ_{N+1} approximation but with $0.5(\xi_{N+1} + \xi_N)$.

By a combination of the method of successive intervals and a continuation of solutions into the region of constant dimensions, it has been found possible to solve a problem of the Stefan type for an arbitrary initial distribution and variable boundary conditions of the second kind. A Stefan problem with variable boundary conditions of the first kind can be solved analogously.

A solution of the type of (13) is a solution for the so-called incorrectly stated problem. It is stable for positions of the boundary $s(\omega)$ satisfying the condition

$$s_N^2/2R^2 \leqslant \Delta\omega \tag{25}$$

In the case of the presence of two fluxes $Q_{1,N+1}$ and $q_{2,N+1}$, the algorithm for the calculation is stable with

$$\Delta\omega = \min\{(R-x)^2/2R^2,\ x^2/2R^2\} \tag{26}$$

For the deepest points, when the selected interval $\Delta\omega$ does not satisfy condition (26), the fluxes $Q_{1,i+1}$ and $q_{2,i+1}$ can be determined using the algorithm of E. M. Sparrow [5, 6] or the algorithm of A. N. Tikhonov [7, 8].

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